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Computational and Implementation Problems in Option Pricing with Long Maturity

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Abstract – In this paper we explore the significance of maturity parameter when the Fourier method is applied in option pricing by using two different analytical solutions of the Black-Scholes equation, that are expressed as infinite series of reflections and Fourier series. Even for fixed barriers contracts different solutions give the same answer when all the terms have been added up but the rate of convergence of the sum to the solution can be quite different, depending on the time to expiry (maturity), i.e. the duration of the contract. Thus, we demonstrate the significance of the maturity parameter in option contracts when analytical methods like Fourier one is applied not only in jump-diffusion models.

Keywords – Furrier transformation and coefficients, Black-Scholes PDE, maturity size parameter, discrete barrier options.

1. Introduction

Options are generally defined as a 'contract' between two parties in which one party has the right but not the obligation to do something in a future moment, usually to buy or sell shares of some commodity or stock. The essential of options is that having rights without obligations has financial value because this deal has no downside (risk) for the holder, i.e. he either makes a profit or quits without a loss. Thus, he has a potential gain, if he manages to foresee the rising or falling values of the underlying asset buying shares, or otherwise he does not exercise this option. To enter in such contract without obligations, the holder should pay some money in advance, i.e. the option price or often known as *premium*.

In the market of financial derivatives, the most important problem is the so-called option valuation problem, i.e. to compute a fair value for the option, i.e. the premium. Usually in financial literature, as a mathematical model for the movement of the asset price St under risk-neutral measure is considered a standard geometric Brownian motion diffusion process with constant coefficients r (interest rate), q (constant dividend) and σ (volatility), i.e. satisfies the following stochastic differential equation:

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t,$$

where dW_t are the increments of Gauss-Wiener process. By the fundamental Itô's lemma [3] for stochastic equations, the following linear parabolic partial differential equation with non-constant coefficients is derived

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0,$$

where t is the current time and the value of the option price is $V(S,t)$ at time t before the expiry T. The option price $V(S,t)$ is specified as a solution of this equation according to the boundary conditions that are used. The last equation is known in literature as the Black-Scholes partial differential equation. It has many solutions, corresponding to all the different options contracts that can be defined with the underlying

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variable *S* and initial and boundary conditions. A very interesting example of boundary conditions is:

Definition: *A discrete double barrier knock-out call option is an option with a payoff condition equal to max(S-K,0) which expires worthless if before the maturity the asset price has fallen outside the barrier corridor [L,U] at the prefixed monitoring dates: at these dates the option becomes zero if the asset falls out of the corridor. If one of the barriers is touched by the asset price at the prefixed dates then the option is canceled, i.e. it becomes zero, but the holder may be compensated by a rebate payment.*

Here are the initial (actual payoff) and boundary conditions of the respective Black-Scholes partial differential equation in case of a discrete double barrier knock-out call options:

$$V(S,0) = max(S-K,0)1_{[L,U]}(S),$$

$$V(S,t) \to 0 \text{ when } S \to 0 \text{ or } S \to \infty,$$

with updating the initial conditions at the monitoring dates $t_{i,}$ $i=1,\ldots,F$:

$$V(S,t_i^+) = V(S,t_i^-)1_{[L,U]}(S),$$

for $0 = t_0 < t_1 < \cdots < t_F = T$, where $1_{[L,U]}(S)$ is the indicator function, i.e. $1_{[L,U]}(S) = 1$ if $S \in [L,U]$ and $1_{[L,U]}(S) = 0$ otherwise. It should be noted that away from the monitoring dates, the option price can move on the positive real axis interval $[0,+\infty)$. If there is a *rebate payment*, i.e. the holder is compensated with a fixed amount if one of the barriers L or U is touched before maturity, then we have additional conditions at the monitoring dates t_i, $i=1,\ldots,F$: $V(L,t_i) = R_L$ and $V(U,t_i) = R_U$.

The Black-Scholes equation is usually transformed to a *diffusion* equation or a heat equation with constant coefficients:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 V}{\partial u^2}, \tau \geq 0,$$

with the following transformations:

$$S = Ke^u, t = T - \frac{2\tau}{\sigma^2}, \rho = \frac{2(r-q)}{\sigma^2}$$

$$V(S,t) = V(Ke^{\,u}, T - \frac{2\tau}{\sigma^2}) =: K\tilde{V}(u,\tau)$$

$$\tilde{V}(u,\tau) = e^{-\frac{(\rho-1)}{2}u - \frac{(\rho+1)^2}{4}\tau - \frac{2q}{\sigma^2}\tau} v(u,\tau)$$

One important problem when transforming the Black-Scholes equation into a heat equation is what happens with the payoff condition *max(S-K,0)* applying the upper transformations? The payoff condition becomes an initial condition *v(u,0)* for the heat equation

$$V(S,t) = \max(S(T)-K,0) = Kmax(e^{\,u}-1,0)$$

And having in mind (10) we obtain *v(u,0)*, i.e.

$$v(u,0) = \max\,(e^{\frac{(\rho+1)}{2}u} - e^{\frac{(\rho-1)}{2}u}, 0).$$

In the simplest case, if a constant barrier value *B* is *continuously* applied, R_b is the rebate received if the barrier is ever breached, the boundary conditions are

$$V(B,t) = R_b,$$

$$v(u_b,\tau) = \frac{R_b}{K} e^{-\frac{(\rho-1)}{2}u - \frac{(\rho+1)^2}{4}\tau - \frac{2q}{\sigma^2}\tau},$$

$$u_b = \ln\,(B/K)$$

And according to the used boundary and payoff (initial) conditions for the different option contracts we could either solve directly the heat equation *analytically* or approximate its solution *numerically* using for example finite difference methods [6].

In absence of explicit formulas numerical methods such as the binomial trees, Monte Carlo simulations and finite difference schemes have been frequently used in option pricing [3].

However, in the last twenty year the application of the Fourier method and Fast-Fourier Transform (FFT) is preferred for the valuation of option price that could be defined as a solution of the Black-Scholes partial differential equation (PDE). The considerable computational power of the FFT and even the most fundamental advances in scientific computing are inefficient when additional financial features such as barriers, dividends and rebate payments are incorporated in option pricing. Furthermore, though the decomposition of an option price into probability elements is theoretically attractive as explained in but it is numerically undesirable due to *discontinuity of the payoffs* [2], [5]. And it comes from the definition for a double barrier knock-out option that is a plain vanilla option but it is knocked out the moment the asset reaches either the upper barrier or the lower barrier. The complicating factor is the *presence of two barriers* and *reflection technique* no longer works: we can use it to satisfy the knock-out condition on one barrier or the other, but not on both simultaneously, as the reflection principle states that the probability of hitting the barrier is exactly twice the probability of ending up below the barrier at expiration. Fortunately, using definition for barrier options that a standard vanilla contract with payoff *max(S-K,0)* is obtained if none of the barriers is touched, the solution can be

found in *two different forms*, each written as an infinite series as it can be seen in Section 2.

2. Option pricing and Fourier solutions of the Black-Scholes PDE

The first form solution for pricing a barrier call option is essentially an *infinite series of reflections*, alternately in the lower and upper barriers. In option pricing most payoffs can be represented as a sum of plain vanilla payoffs [1], [3];

The second form is that of a *Fourier series*, *an infinite sum of special solutions* which satisfy the boundary conditions, and the payoff condition is satisfied by taking the correct combination. Both solutions give the same answer when all the terms have been added up, but as it is shown by Milev in [4] that the *rate of convergence of* the sum to the solution can be quite different, depending on the *time to expiry*. Each of the above-described solutions, i.e. *infinite series of reflections* and *Fourier series one*, has the following main distinctive features, respectively:

As the *number of reflections increases*, the *'images'* get further and further away from the original strip, and their contribution to the option value becomes correspondingly *smaller*, see the analysis below. Even in terms of the heat equation the coefficients are not easy to evaluate, and it may be better to calculate them numerically using for example Simpson's rule. But in contrast to the reflection solution, the Fourier series solution converges very badly when *t* is close to maturity *T*. It is easiest to explain these solutions in the heat equation framework. For the first, we use that a reflection for Black-Scholes equation correspond to translation by a constant for the heat equation. For the Fourier series it is much easier to calculate the special solutions for the simpler heat equation.

The series solution is clearly preferable to the Fourier series version in case of *small* τ , i.e. near the expiry *T,* where the direction of the time variable t is usually changed by $t = T - \tau, \tau \in [0, T]$. But one serious drawback of the solution is that as *the number of reflections increases*, the *'images'* get further and further away from the original strip, and their contribution to the option value becomes correspondingly *smaller*. This effect is particularly observed for *larger* τ , i.e. smaller values of time *t*. In such cases the Fourier series converges much more rapidly since its latter terms

$$e^{-\frac{\pi^2\sigma^2}{2\ln^2(U/L)}n^2(T-t)}$$

decay rapidly with increasing the time to expiry in the following option pricing formula *V(S,t)*

$$V(S,t) = S^k e^{\frac{1}{2}\sigma^2\lambda(T-t)} \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi(\ln(S/L)}{\ln(U/L)}\right) e^{\left(-\frac{\pi^2\sigma^2n^2(T-t)}{2\ln^2(U/L)}\right)} \quad (1)$$

where:

$$k = -\frac{1}{2}(\rho - 1), \lambda = -\frac{1}{4}(\rho + 1)^2, \rho = \frac{2r}{\sigma^2}$$

and the Fourier coefficients α_n are:

$$\alpha_n = \frac{2}{\ln(U/L)} \int_L^U S^{-k-1} \Lambda(S) \sin\left(\frac{n\pi(\ln(S/L)}{\ln(U/L)}\right) \quad (2)$$

where $\Lambda(S)$ is the payoff condition, i.e.

$$\Lambda(S) = \max(S - K, 0) \text{ and } \Lambda(S) = \max(K - S, 0)$$

for call and put option, respectively.

3. Numerical Discussion and Significance of the Maturity Length

It is well-known that *the reflection solution* should always be used close to expiry *T*, because its terms are customized to the discontinuities of the payoff [1], [4]. Conversely, the Fourier series converges well for times *t long before expiry*, because of the rapid time-decay of the exponential coefficients. It also converges better for *larger volatility parameter* σ. Unfortunately, the Fourier series solution converges very badly when *t* is close to *T* and when σ takes low values. i.e. well-known problem in Quantitative Finance known as *low volatility problem* [3], [6], [9]. The Fourier coefficients α_n in (2) should be calculated numerically.

But the restrictions of the described above analytical solutions consist not only of the *convergence rate dependence on time to maturity* but also of the difficult generalizations for financial contracts in real life. For example, we have, $\rho = \frac{2r}{\sigma^2}$ and if there are *continuous yield dividend D* then

$$\rho = \frac{2(r - D)}{\sigma^2}.$$

But the trading in real life is done in *discrete moments*, i.e. monthly, weekly or daily, and image and Fourier series solutions should be adapted for *discrete proportional dividends* and *discrete barriers*. This problem is easier to be resolved by *semi-analytical numerical methods* [5], [7], [10] or *traditional Quantitative methods* such as the well-known Monte Carlo simulation [2] and classical finite difference schemes [2], [6].

In case of *Lévy processes* where the *size of the maturity* parameter is important some existing algorithms such as this of Milev-Tagliani [5] are extended [10]. And it is necessary more sophisticated application of the classical Fourier method such as a Fast Fourier transform [2]. As shown in Chap. 3, any application of Fourier methods leads to the evaluation of a discrete Fourier transform of length N (DFT\((N)\)). Thus, the efficient computation of DFT\((N)\) is very important. A FFT is an algorithm for computing the DFT\((N)\) which needs only *a relatively low number of arithmetic operations (N),* [8]. In case of different than the lognormal processes such as these of Merton and Kou, Phelana et. al. in [7] present the characteristic exponent of some Lévy processes and discrete Fourier and Hilbert transforms for pricing barrier options.

When other path-dependent contracts are valuated most of the existing methods for pricing Asian options are less efficient in the limit of small maturities and small volatilities [9].

4. Conclusions

The application of the analytical Fourier method is often preferred in option valuation because finding the solution to the Black-Scholes equation is a trivial mathematical task, but the numerical calculation is inefficient, which is a disadvantage in contracts with small maturities/small volatilities or multi-asset pricing. The flexibility of any analytical or numerical method in terms of achieving both efficient computational/real times makes it very competitive with the most frequently used numerical methods in Quantitative Finance that have a heuristic origin, i.e. the binomial trees and the Monte Carlo simulations. But the classic parametrical models often suffer from several limitations in term of computational power required for parametric calibration and unrealistic economical and statistical assumptions [12].

A key advantage of recent computational methods is also simplicity of their implementation that includes many financial parameters such as discrete barriers, dividends and rebate payments as well as early exercise possibilities that could be directly incorporated at any moment during the life of the option and for every value of underlying asset price. Such features are indispensable for option contracts traded in real life but including all of them together is difficult for most numerical methods.

Probably the future advances in option pricing are artificial neural networks (ANN) where unlike the traditional methods, the neural network does not care about the specific meaning of the input parameters, but the degree of its influence on the model [11]. ANN and its derivatives are just all black boxes.

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